ICOM 4075 FALL 2014

Homework Assignment #3 SOLUTIONS.

1.6.4 (c) Modus Ponens

(d) Addition

(e) Hypothetical Syllogism

1.6.6 We use the following propositions

$p1 \sim$ it rains

$p2 \sim$ it is foggy

$p3 \sim$ race is held

$p4 \sim$ lifesaving demo held

$p5 \sim$ trophy awarded

Our premises are :

$P_6 \sim (\neg p_1 \vee \neg p_2) \to (p_3 \wedge p_4)$

$P_7 \sim P_3 \to P_5$

$P_8 \sim \neg P_5$

Applying modus tollens to P_8 and P_7 we infer $\neg P_3$

$\neg P_3$ allows us to infer $\neg(P_3 \wedge P_4)$

Modus tollens applied to P_6 and $\neg(P_3 \wedge P_4)$ yields $\neg(\neg P_1 \vee \neg P_2)$

Applying de Morgan's we get $P_1 \wedge P_2$

And by Simplification $\underline{P_1}$

Therefore it rained.

1.6.10

(d) domain : people.

$A(x) \sim x$ has an accont

$S(x) \sim x$ is a student.

Premises:

Every student has an Internet accont : $\forall_x (S(x) \to A(x))$

Homer does not have Internet accont : $\neg A(\text{Homer})$

Conclusions :

$S(\text{Homer}) \to A(\text{Homer})$ Universal Inst.

$\neg S(\text{Homer})$ Modus tollens.

Homer ~~does~~ is not a student. .

Can't tell wether Maggie is or not a student.

(e) domain: foods.

Predicates :

$H(f) \sim f$ is healthy food

$T(f) \sim f$ tastes good

$M(f) \sim$ I eat f.

Premises :

All healthy foods do not taste good $\sim$ $\forall_f\, H(f) \to \neg T(f)$

Tofu is healthy $\sim$ $H(\text{Tofu})$

~~You~~ I only eat what tastes good $\sim$ $\forall f\; M(f) \to T(f)$

Cheesburgers are not healthy $\sim$ $\neg H(\text{Cheesburgers})$.

Conclusions

$H(\text{Tofu}) \to \neg T(\text{Tofu})$ Universal Inst.

$\neg T(\text{Tofu})$ Modus Ponens

Tofu does not taste good.

$M(\text{Tofu}) \to T(\text{Tofu})$ Univ. Inst.

$\neg M(\text{Tofu})$ Modus Tollens.

I donot eat Tofu.

1.6.10 Continued...

(f) d ~ I am dreaming
h ~ I am hallucinating
e ~ I see elephants.

Premises:

p_1 ~ I am either dreaming or hallucinating ~ $d \vee h$

p_2 ~ I am not dreaming ~ $\neg d$.

p_3 ~ If I am hallucinating I see elephants ~ $h \to e$.

Conclusions:

Applying disjunctive syllogism to p_1 and p_2 we infer h.

Applying modus ponens to h and p_3 we infer e.

~~We~~ I see elephants running down the road!

1.6.12 Show
$$\begin{array}{l} (p \wedge t) \to (r \vee s) \\ q \to (u \wedge t) \\ u \to p \\ \neg s \\ \hline q \to r \end{array}$$
is a valid argument.

Applying the conclusion of exercise 1.6.11 it will suffice to show

p_1: $(p \wedge t) \to (r \vee s)$
p_2: $q \to (u \wedge t)$
p_3: $u \to p$
p_4: $\neg s$
p_5: q
c: r

1. q premise
2. $u \wedge t$ modus ponens to p_2
3. u simplification of 2
4. p modus ponens to p_3
5. t simplification of 2
6. $p \wedge t$ conjunction of 4 and 5.
7. $r \vee s$ modus ponens p_1
8. $\neg s$ premise
9. r disjunctive syllogism 7 & 8

1.6.14

(a) Predicates:

$S(x)$ ~ x is a student in class

$C(x)$ ~ x owns a red convertible

$T(x)$ ~ x has gotten a ticket.

Premises

$S(\text{Linda})$: Linda is a student

$C(\text{Linda})$: Linda owns a red convertible

$\forall x(C(x) \to T(x))$: Everyone who own a red convertible has gotten ticket

Conclusion

$\exists x \; S(x) \land T(x)$

Valid Argument:

1. $S(\text{Linda})$ premise
2. $C(\text{Linda})$ premise

~~3. S(Linda~~

3. $\forall x\,(C(x) \to T(x))$ premise
4. $C(\text{Linda}) \to T(\text{Linda})$ Universal Instantiation 3
5. $T(\text{Linda})$ Modus ponens 2 & 4
6. $\exists x \; T(x)$ Existential generalization.

1.6.14 Continued...

(b) Predicates domain: students.

$D(x)$ ~ x has taken discrete structures.

$A(x)$ ~ x can take algorithms.

Premises

$D(s)$ for s each one of the five students.

$\forall x (D(x) \to A(x))$ Every student who has taken DS can take algorithms

For each student s the argument is as follows:

1. $D(s)$ premise
2. $\forall x (D(x) \to A(x))$ premise
3. $D(s) \to A(s)$ Universal Instantiation of 2
4. $A(s)$ Modus ponens 1 & 3.

student s can take algorithms.

(c) domain: movies

Predicates

$J(x)$ ~ John Sayles produced x.

$W(x)$ ~ x is wonderful.

Premises:

$\forall x (J(x) \to W(x))$ ~ all movies by Sayles are wonderful

J(movie about coal miners)

Valid Argument:

1. J(movie about coal miners) — MACM premise
2. $\forall x (J(x) \to W(x))$ premise
3. $J(MACM) \to W(MACM)$ Universal Inst.
4. $W(MACM)$ Modus Ponens
5. $\exists x\ W(x)$ Existential generalization.

1.6.20

(a) Not valid

(b) Valid

1.6.26

Premises:

$p_1: \forall x\, (P(x) \to Q(x))$

$p_2: \forall x\, (Q(x) \to R(x))$

Conclusion

$\forall x\, (P(x) \to R(x))$

→ Valid argument

1.	$\forall x\, (P(x) \to Q(x))$	premise
2.	$\forall x\, (Q(x) \to R(x))$	premise
3.	Let a be an arbitrary element of the domain	premise

~~4. P(a) →~~

Case I $\neg P(a)$

Then $P(a) \to R(a)$ vacuously true

Case II $P(a)$ Premise

a.	$P(a) \to Q(a)$	Univ. Inst. 1.
b.	$Q(a)$	Modus Ponens
c.	$Q(a) \to R(a)$	Univ Inst 2.
d.	$R(a)$.	Modus Ponens
e.	$P(a) \to R(a)$	

1.6.28

Premises:

$\forall x\; P(x) \vee Q(x)$

$\forall x\; (\neg P(x) \wedge Q(x)) \to R(x)$

Must show:

$\forall x\, (\neg R(x) \to P(x))$

Consider an arbitrary element a from the domain

We will consider two cases $R(a)$ and $\neg R(a)$

Case I $\neg R(a)$

$\neg P(a) \wedge Q(a) \to R(a)$	
$\neg(\neg P(a) \wedge Q(a))$	Mod. Toll.
$P(a) \vee \neg Q(a)$	De Morgan's
$P(a) \vee Q(a)$	Univ. Inst.
$P(a)$	Resolution
$\neg R(a) \to P(a)$	
$\forall x\, \neg R(x) \to P(x)$	Univ. Gen.

Case II $R(a)$

$\neg R(a) \to P(a)$ vacuously true since $\neg R(a)$ false.

1.7.6 Show the product of two odd numbers is odd.

Direct Proof.

Let x and y two arbitrary odd numbers.

$x = 2k_1 + 1$ and $y = 2k_2 + 1$ for some integers k_1, k_2

$$x \cdot y = (2k_1+1)(2k_2+1) = 4k_1k_2 + 2k_1 + 2k_2 + 1$$
$$= 2(2k_1k_2 + k_1 + k_2) + 1$$

$x \cdot y$ is odd.

1.7.8 Show n perfect square $\rightarrow$ n+2 is not perfect square.

Proof by Contradiction

Let n be an arbitrary integer.

Assume n is a perfect square and n+2 is also a perfect square.

$n = a^2$ and $n+2 = b^2$ for integers a, b.

$$b^2 - a^2 = n+2-n = 2$$

$$(b+a)(b-a) = 2$$

But since 2 is a prime and $b > a$

The only possible solution is $b+a = 2$ and $b-a = 1$

This system has no solution.

A contradiction.

(1.7.12) Prove or disprove ~~Show~~ that the product of a nonzero rational number and an irrational number is irrational.

We will prove that this is true. by contradiction.

Let r_1 a nonzero rational and i an irrational.

Assume $r_1 \cdot i$ is a rational number. r_2.

$r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$ for integers a, b, c, d.

$r_1 \cdot i = r_2$ implies $\frac{a}{b} \cdot i = \frac{c}{d}$

But then $i = \frac{c}{d} \cdot \frac{b}{a} = \frac{cb}{da}$

i is rational A contradiction!

(1.7.18) Prove $3n+2$ even $\rightarrow$ n even.

(a) By contraposition.

Assume n odd

$n = 2k+1$ for some integer k.

$3n+2 = 3(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1$

$3n+2$ odd.

(b) By contradiction.

Assume $3n+2$ even $\wedge$ n odd.

$n = 2k+1$ for some k integer

$3n+2 = 3(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1$

$3n+2$ odd

« Contradiction

1.7.20 $P(n)$: n a positive integer $\rightarrow n^2 \geq 1$

$P(1)$ 1 a positive integer $\rightarrow (1)^2 \geq 1$

Proof Direct.

1 is a positive integer.

$(1)^2 = 1 \geq 1$

Thus $P(1)$

1.7.28 Prove $m^2 = n^2 \leftrightarrow m = n$ or $m = -n$.

$(\rightarrow)$ Show $m^2 = n^2 \rightarrow m = n$ or $m = -n$.

Proof by contraposition

Assume $\neg(m = n$ or $m = -n)$

By DeMorgan's $(m \neq n \wedge m \neq -n)$

Case I $(m = 0 \vee n = 0)$

$m^2 = 0$

Thus $n^2 = 0$ and $n = 0$

And $m = 0$

Therefore $m = n$.

Case II $m \neq 0$ and $n \neq 0$

$m^2 = n^2$

Then $\frac{m^2}{n^2} = 1 = \left(\frac{m}{n}\right)^2$

Thus $\frac{m}{n} = 1$ or $\frac{m}{n} = -1$

And then $m = n$ or $m = -n$.

$(\leftarrow)$ Assume $m = n$ or $m = -n$

$m^2 = (n)^2$ or $m^2 = (-n)^2$

$m^2 = n^2$, $m^2 = n^2$

Thus $m^2 = n^2$.

1.7.42 Prove equivalence of these statements

$p1$: n^2 is odd
$p2$: $1-n$ even
$p3$: n^3 odd
$p4$: n^2+1 even

Strategy:
Prove $\to p1 \to p2 \to p3 \to p4$

($p1 \to p2$) n^2 odd $\to$ $1-n$ even
By Contraposition.
Assume $1-n$ odd
$1-n = 2k+1$ for some integer k.
$1-2k-1 = n$
$-2k = n$
$2(-k) = n$
n even
n^2 even

($p2 \to p3$) $(1-n)$ even $\to$ n^3 odd
Direct Proof
Assume $1-n$ even
$1-n = 2k \to n = 1-2k$

$$n^3 = (1-2k)^3 = (1-2k)(1-4k+4k^2) = 1-4k+4k^2-2k+8k^2-8k^3$$
$$= 1-6k+12k^2-8k^3$$
$$= 2(-3k+6k^2-4k^3)+1$$

n^3 odd.

($p3 \to p4$) n^3 odd $\to$ n^2+1 even
Assume n^2+1 odd
$n^2+1 = 2k+1$ for some $k \in \mathbb{Z}$
$n^2 = 2k$
n even (theorem shown in class)
$n = 2k_2$
$n^3 = 8k_2^3$
n^3 even

next page $\to$

1.7.42 Continued

$(p_4 \to p_1)$ n^2+1 even $\to$ n^2 odd.

Assume n^2 even

Then n even (Theorem ~~shown~~ proved in class)

$n = 2k$ for some $k \in \mathbb{Z}$.

$n^2+1 = (2k)^2+1 = 4k^2+1 = 2(2k^2)+1$

n^2+1 is odd.

1.8.2 There are no positive perfect cubes less than 1000 that are the sum of two positive perfect cubes

Exhaustive Proof

The set of positive perfect cubes is $\{1, 8, 27, 64, 125, 216, 343, 512, 729\}$ ~~1000~~

By simple inspection no such number is the sum of two other members of the set.

1.8.6 $5x+5y$ is odd $\leftarrow$ x, y have different parity.

There are two possibilities (x odd $\wedge$ y even) or (x even $\wedge$ y odd)

WLG lets assume x odd $\wedge$ y even. (The other proof would be identical)

Then $x = 2k_1+1$ and $y = 2k_2$ for $k_1, k_2 \in \mathbb{Z}$

$5x+5y = 5(x+y) = 5(2k_1+1+2k) = 2(5k_1+5k_2)+5 = 2(5k_1+5k_2+2)+1$

$5x+5y$ is odd.

1.8.8 Must show $\exists n$ $1+2+3+\ldots+n = n$ $\wedge$ n positive.

Let $n=1$

n is a positive integer.

integers not exceeding $1 = \{1\}$

Their sum is 1.

Q.E.D.

1.8.18

Let i be an arbitrary irrational

Then i is also real.

Therefore $i = n + \epsilon$ $0 \leq \epsilon < 1$.

Moreover $\epsilon \neq 0$, since i is not rational.

We will consider two cases separately.

Case I $0 < \epsilon < \frac{1}{2}$ (ϵ cannot be $1/2$ since i is irrational)

Let $m = n$ (from $i = n + \epsilon$)

m is an integer

$x - n = \epsilon$ which is less than $1/2$.

Case II $\frac{1}{2} < \epsilon < 1$

Let $m = n+1$ which is also in $\mathbb{Z}$.

The distance from i $= |n+1-i| = |\cancel{n}+1-(\cancel{n}+\epsilon)| = 1-\epsilon$

Since $\frac{1}{2} < \epsilon < 1$ then $0 < 1-\epsilon < \frac{1}{2}$

This m is a valid integer that satisfies the theorem.

Need to prove uniqueness now

next page $\longrightarrow$

1.8.12

Base	$L = \log_2 base$	exp	$L * exp$	floor ceiling		
65	6.022...	1000	6022	floor		
8	3	2001	6003	ceiling		positive
3	1.58...	177	280	floor		
79	6.30...	1212	7640	floor		
9	3.16...	2399	7405	ceiling		positive
2	1	2001	2001	floor		
24	4.58...	4493	20600	floor		
5	2.32...	8192	19022	ceiling		positive.
7	2.80...	1777	4988	floor		

Notes:

We convert all numbers to the form 2^k by taking the $\log_2$ of their bases. We then multiply the $\log_2$ by the exponent and take the ceiling for negative numbers and the floor for positive numbers.

In all cases, even reducing positives and increasing negatives we get positives.

Thus the product of any three numbers is positive.

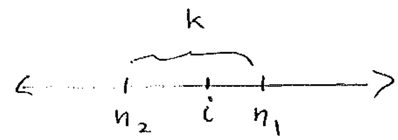
1.8.18 Continued

Suppose there are two distinct integers n_1, n_2 such that $|i - n_1| < \frac{1}{2}$ and $|i - n_2| < \frac{1}{2}$

WLG let's assume $n_1 > n_2$

Then $n_1 = n_2 + k$ for some k in $\mathbb{Z}^+$ $k \geq 1$

Since $|i - n_1| < \frac{1}{2}$



$$-\frac{1}{2} < r - n_1 < \frac{1}{2}$$

$$-\frac{1}{2} < r - (n_2 + k) < \frac{1}{2}$$

$$-\frac{1}{2} < r - n_2 - k < \frac{1}{2}$$

$$-\frac{1}{2} + k < r - n_2 < \frac{1}{2} + k.$$

$$\frac{1}{2} < r - n_2 < \frac{1}{2} + k$$

A contradiction.

1.8.22 Prove $\forall x \in \mathbb{R}$ $x \neq 0 \rightarrow x^2 + \frac{1}{x^2} \geq 2$

Assume $x \neq 0$ where x is an otherwise arbitrary real.

The hint tells use that $(x - \frac{1}{x})^2 \geq 0$ holds.

Expanding the binomial, $x^2 - 2\frac{x}{x} + \frac{1}{x^2} \geq 0$

$$x^2 - 2 + \frac{1}{x^2} \geq 0$$

$$x^2 + \frac{1}{x_2} \geq 2$$

Q.E.D.