Section 2.1

(10) a - T b - T c - F d - T e - T f - T

g - T

(22) $P(A) = P(B) \xrightarrow{?} A = B$

Solution: $x \in A \to \{x\} \in P(A) \to \{x\} \in P(B) \to x \in B$

$x \in B \to \{x\} \in P(B) \to \{x\} \in P(A) \to x \in A$

$A \subset B \wedge B \subset A \to A = B$

(24) A Power set must have 2^n elements for an $n \in \mathbb{N}$

Thus c is not a power set because it has 3 elements. Any Power set must have $\emptyset$ as a member

So a is not a power set since $\emptyset \notin \emptyset$.

$b = P(\{a\})$ and $d = P(\{a,b\})$ so They are Power sets.

(26) $A \subseteq C \wedge B \subseteq D \xrightarrow{?} A \times B \subseteq C \times D$

$x \in A \times B \to x = (a,b) \wedge a \in A \wedge b \in B$

$a \in A \wedge A \subseteq C \to a \in C$ $b \in B \wedge B \subseteq D \to b \in D$

$x = (a,b) \wedge a \in C \wedge b \in D \to (a,b) \in C \times D \to x \in C \times D$

$(x \in A \times B \to x \in C \times D) =$ implies $A \times B \subseteq C \times D$

(34) a) $A^3 = \{a\} \times \{a\} \times \{a\} = \{(a,a,a)\}$

b) $A^3 = \{0,a\} \times \{0,a\} \times \{0,a\}$

$= \{(0,0,0), (a,0,0), (0,a,0), (0,0,a), (0,a,a)$

$, (a,0,a), (a,a,0), (a,a,a)\}$

(38) $A \neq B \xrightarrow{?} A\times B \neq B \times A$

$A \neq B \rightarrow \exists x \in A\ x \notin B \ \vee\ \exists x \in B\ x \notin A$

without loss of generality assume $x_0 \in A$ and $x_0 \notin B$

So There would be at least one pair (x_0, b) $b \in B$ in

$A\times B$ while There is no any pair with The first element

equal to x_0 in $B\times A$. That means $(x_0, b) \notin B\times A$

having $(x_0,b) \in A\times B \wedge (x_0,b) \notin B\times A$ implies $A\times B \neq B\times A$

Section 2.2

(10) $A - \emptyset = A$

$A - \emptyset = A \cap \overline{\emptyset} = A \cap U = A$

$\emptyset - A = \emptyset$

$\emptyset - A = \emptyset \cap \overline{A} = \emptyset$

(14) $A = \{1, 5, 7, 8, 3, 6, 9\}$

$B = \{2, 10, 3, 6, 9\}$

approach: everything in $A-B$ is in A

everything in $B-A$ is in B

everything in $A \cap B$ is in boath

(16) d) $A \cap (B-A) = \emptyset$

$x \in A \cap (B-A) \rightarrow x \in A \wedge x \in B-A \rightarrow x \in A \wedge x \in B \wedge x \notin A$

$\rightarrow x \in A \wedge x \in \bar{A} \wedge x \in B \rightarrow x \in A \cap \bar{A} \wedge x \in B \rightarrow x \in \emptyset \wedge x \in B$

$\rightarrow x \in \emptyset$

$x \in \emptyset \rightarrow x \in A \cap (B-A)$ Vacuous proof.

e) $A \cup (B-A) = A \cup B$

$A \cup (B-A) = \{x \mid x \in A \vee x \in B-A\} = \{x \mid x \in A \vee (x \in B \wedge x \notin A)\}$

$= \{x \mid x \in A \vee (x \in B \wedge x \in \bar{A})\}$.

$= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in \bar{A})\}$

$= \{x \mid x \in A \cup B \wedge x \in A \cup \bar{A}\} = \{x \mid x \in A \cup B \wedge x \in U\}$

$= \{x \mid x \in (A \cup B) \cap U\} = \{x \mid x \in A \cup B\} = A \cup B$

(18) e) $(B-A)\cup(C-A) = (B\cup C)-A$

A	B	C	B-A	C-A	(B-A)∪(C-A)	B∪C	(B∪C)-A
1	1	1	0	0	0	1	0
1	1	0	0	0	0	1	0
1	0	1	0	0	0	1	0
1	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1
0	0	1	0	1	1	1	1
0	0	0	0	0	0	0	0

b) $(A-C)\cap(C-B)=\phi$

$(A-C)\cap(C-B) = \{x \mid x\in A-C \wedge x\in C-B\}$

$= \{x \mid x\in A \wedge x\in \bar{C} \wedge x\in C \wedge x\in \bar{B}\}$

$= \{x \mid x\in \bar{C} \wedge x\in C \wedge x\in A \wedge x\in \bar{B}\}$

$= \{x \mid x\in (C\cap\bar{C}) \wedge x\in A \wedge x\in \bar{B}\}$

$= \{x \mid x\in \phi \wedge x\in A \wedge x\in \bar{B}\} = \phi$

(36) $A \oplus B = (A \cup B) - (A \cap B)$

one way

$x \in A \oplus B \rightarrow x \in \big((A-B) \cup (B-A)\big) \rightarrow x \in A-B \vee x \in B-A$

$\rightarrow (x \in A \wedge x \in \bar{B}) \vee (x \in B \wedge x \in \bar{A})$

$\rightarrow \big((x \in A \wedge x \in \bar{B}) \vee x \in B\big) \wedge \big((x \in A \wedge x \in \bar{B}) \vee x \in \bar{A}\big)$

$\rightarrow \big((x \in A \vee x \in B) \wedge (x \in B \vee x \in \bar{B})\big) \wedge \big((x \in A \vee x \in \bar{A}) \wedge (x \in \bar{A} \vee x \in \bar{B})\big)$

$\rightarrow (x \in A \cup B \wedge x \in B \cup \bar{B}) \wedge (x \in A \cup \bar{A} \wedge x \in \bar{A} \cup \bar{B})$

$\rightarrow (x \in A \cup B \wedge x \in U) \wedge (x \in U \wedge x \in \overline{A \cap B})$

$\rightarrow x \in (A \cup B) \cap U \wedge x \in U \cap \overline{A \cap B}$

$\rightarrow x \in A \cup B \wedge x \in \overline{A \cap B} \rightarrow x \in (A \cup B) \cap \overline{(A \cap B)}$

$\rightarrow x \in (A \cup B) - (A \cap B)$ ~~Therefore~~ $\boxed{A \oplus B \subseteq (A \cup B) - (A \cap B)}$

another way

$x \in (A \cup B) - (A \cap B) \rightarrow x \in (A \cup B) \cap \overline{(A \cap B)}$

$\rightarrow (x \in A \vee x \in B) \wedge (x \in \overline{A \cap B})$

$\rightarrow (x \in A \vee x \in B) \wedge (x \in \bar{A} \cup \bar{B}) \rightarrow (x \in A \vee x \in B) \wedge (x \in \bar{A} \vee x \in \bar{B})$

$\rightarrow \big((x \in A \vee x \in B) \wedge (x \in \bar{A})\big) \vee \big((x \in A \vee x \in B) \wedge x \in \bar{B}\big)$

$\rightarrow (x \in A \wedge x \in \bar{A}) \vee (x \in B \wedge x \in \bar{A}) \vee (x \in A \wedge x \in \bar{B}) \vee (x \in B \wedge x \in \bar{B})$

$\rightarrow \underbrace{\cancel{x \in \emptyset}}_{F} \vee x \in B-A \vee x \in A-\bar{B} \vee \underbrace{\cancel{x \in \emptyset}}_{F} \rightarrow$ $\boxed{\big((A \cup B) - (A \cap B)\big) \subseteq A \oplus B}$

$x \in B-A \vee x \in A-B \rightarrow x \in (A-B) \cup (B-A) \rightarrow x \in A \oplus B$ thus

Continued

(36) Set builder:

$$A \oplus B = \{x \mid x \in A-B \lor x \in B-A\}$$

$$= \{x \mid (x \in A \land x \in \overline{B}) \lor (x \in B \land x \in \overline{A})\}$$

$$= \{x \mid ((x \in A \land x \in \overline{B}) \lor x \in B) \land ((x \in A \land x \in \overline{B}) \lor x \in \overline{A})\}$$

$$= \{x \mid (x \in A \lor x \in B) \land (x \in B \lor x \in \overline{B}) \land (x \in A \lor x \in \overline{A}) \land (x \in \overline{A} \lor x \in \overline{B})\}$$

$$= \{x \mid x \in A \cup B \land \underbrace{x \in U}_{T} \land \underbrace{x \in U}_{T} \land x \in \overline{A} \cup \overline{B}\}$$

$$= \{x \mid x \in A \cup B \land x \in \overline{A \cap B}\} = \{x \mid x \in (A \cup B) \cap (\overline{A \cap B})\}$$

$$= \{x \mid x \in (A \cup B) - (A \cap B)\} = (A \cup B) - (A \cap B)$$

table

A	B	$A \cup B$	$A \cap B$	$A \cup B - A \cap B$	$A \oplus B$
1	1	1	1	0	0
1	0	1	0	1	1
0	1	1	0	1	1
0	0	0	0	0	0

$\equiv$

(50) c) $A_i = (0, i)$

$$\bigcup_{i=1}^{\infty} A_i = (0,1) \cup (0,2) \cup (0,3) \cup \dots$$

$$= (0, \infty)$$

$$\bigcap_{i=1}^{\infty} A_i = (0,1) \cap (0,2) \cap (0,3) \cap \dots$$

$$= (0,1)$$

d) $A_i = (i, \infty)$

$$\bigcup_{i=1}^{\infty} A_i = (1,\infty) \cup (2,\infty) \cup (3,\infty) \cup \dots$$

$$= (1,\infty)$$

$$\bigcap_{i=1}^{\infty} A_i = (1,\infty) \cap (2,\infty) \cap (3,\infty) \cap \dots$$

$$= \emptyset$$

Note for the last one: if we assume that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$ and $x_0 \in \bigcap_{i=1}^{\infty} A_i$ then $x_0 \notin (\lceil x_0 \rceil, \infty)$ which contradicts the fact that an element in the intersection of some sets belongs to all of them.

Section 2.3

(2) a) No b) Yes c) No

because there is $n \in \mathbb{Z}$ which has more than 1 image. (actually every number in $\mathbb{Z}$ except 0)

Because f doesn't assign anything to 2 and -2 while $2 \in \mathbb{Z}$ and $-2 \in \mathbb{Z}$

(4) a) $D = \mathbb{N}$ $R = \{0,1,2,\dots,9\}$

b) $D = \mathbb{Z}^+$ $R = \mathbb{Z}^- - \{1\}$

d, e) $D =$ Set of all strings over alphabet $\{0,1\}$

$R = \mathbb{N}$

(12) a) yes

$f(n_1) = f(n_2) \rightarrow n_1 - 1 = n_2 - 1 \rightarrow n_1 = n_2$

b) No

$f(5) = f(-5) = 26$

c) Yes

$f(n_1) = f(n_2) \rightarrow n_1^3 = n_2^3 \rightarrow n_1 = n_2$

d) No $\lceil 3/2 \rceil = \lceil 4/2 \rceil = 2$

(14) a) yes

for any $z \in \mathbb{Z}$ we have 2 cases. z is even or odd. Case 1 (even) $z = 2k$:

$(k, 0) \in \mathbb{Z} \times \mathbb{Z}$ -- $f(k, 0) = 2k = z$

case 2 (odd) $z = 2k - 1$

$(k, 1) \in \mathbb{Z} \times \mathbb{Z}$ $f(k, 1) = 2k - 1 = z$

b) No

for $z = 2$ There is no any Pair (m, n) such That $m^2 - n^2 = 2$:

The Perfect squares are as Follows

$(0)^2, (\mp 1)^2, (\mp 2)^2, (\mp 3)^2, \ldots$

The minimum difference of pairs in The list is 1 The next minimum is 3 so There is no any Pair That can make 2.

c) yes

for any $z \in \mathbb{Z}$ $f((z-1, 0)) = z$

d) yes

for any $z \in \mathbb{Z}$ if $z \geq 0$ $f((z, 0)) = z$
if $z < 0$ $f((0, z)) = z$

14 - e) No

For $z=1$ we don't have any $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ s.t. $f(m,n)=1$. Let's say we have and $f(m,n)=1 \rightarrow m^2-4=1 \rightarrow m=\pm\sqrt{5} \notin \mathbb{Z}$ which is contradiction.

(22) a) yes - it is a line.

1-1 : $f(x_1)=f(x_2) \rightarrow -3x_1+4=-3x_2+4 \rightarrow x_1=x_2$

onto : $\cancel{y=} -3x+4 \rightarrow y-4=-3x$

$\rightarrow x=\frac{4-y}{3}$

$\forall y \in \mathbb{R}$ we have $f\left(\frac{4-y}{3}\right)=y$

b) No, it is not 1-1

$f(-1)=f(1)=4$

c) It is not a function from $\mathbb{R}$

let's fix the problem and assume $f: \mathbb{R}-\{-2\} \rightarrow \mathbb{R}$ yet it is not bijection because it is not onto.

for $y=1$ There is no $x \in \mathbb{R}-\{-2\}$ s.t. $f(x)=1$

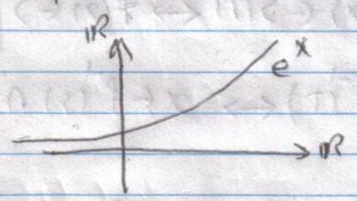
here is how to obtain it: $y=\frac{x+1}{x+2} \rightarrow x=\frac{1-2y}{y-1} \rightarrow y\neq 1$

22-d yes

$f(x_1) = f(x_2) \rightarrow x_1^5 + 1 = x_2^5 + 1 \rightarrow x_1 = x_2$

$y \in \mathbb{R} \rightarrow f(\underbrace{\sqrt[5]{y-1}}_{x}) = y$

(28) way 1 – The following graph represents e^x



The graph shows That e^x is not onto so it cannot be inversible.

if The codomain is $(0, \infty)$ Then it is onto and since e^x is 1-1 it is a bijection Thus it would be inversible.

way 2 – for any x $e^x > 0$ so for $-1 \in \mathbb{R}$ There is no any x s.t. $f(x) = -1$ so e^x is not onto and not inversible. Excluding negative numbers and zero from Codomain makes it onto and Therefore inversible.

(38) $a(cx+d)+b = c(ax+b)+d$

$\leftrightarrow \cancel{acx} + ad + b = \cancel{acx} + cb + d \leftrightarrow ad + b - cb - d = 0$

$\leftrightarrow d(a-1) + b(1-c) = 0 \leftrightarrow d(a-1) = b(c-1)$

$\Longleftrightarrow \dfrac{b}{a-1} = \dfrac{d}{c-1}$

(46) $f: A \to B$ $\quad S, T \subseteq B \quad$ f inversible

a) $x \in f^{-1}(S \cup T) \leftrightarrow f(x) \in S \cup T \leftrightarrow f(x) \in S \vee f(x) \in T$

$\leftrightarrow x \in f^{-1}(S) \vee x \in f^{-1}(T) \leftrightarrow x \in f^{-1}(S) \cup f^{-1}(T)$

b) $x \in f^{-1}(S \cap T) \leftrightarrow f(x) \in S \cap T \leftrightarrow f(x) \in S \wedge f(x) \in T$

$\leftrightarrow x \in f^{-1}(S) \wedge x \in f^{-1}(T) \leftrightarrow x \in f^{-1}(S) \cap f^{-1}(T)$

(48) let's say x is not an integer. That means

$\exists n \in \mathbb{Z}$ and $\exists \varepsilon \in (0,1)$ s.t. $x = n + \varepsilon$

$\lceil x \rceil - \lfloor x \rfloor = \lceil n+\varepsilon \rceil - \lfloor n+\varepsilon \rfloor = (n+1) - (n) = 1$

let's say x is an integer. so $\lceil x \rceil = \lfloor x \rfloor = x$

Therefor $\lceil x \rceil - \lfloor x \rfloor = 0$

(52) a) we know that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$

right to left: assume $\lceil x \rceil \leq n$ since $x \leq \lceil x \rceil$ Then $x \leq \lceil x \rceil \leq n$

There for $x \leq n$

it is continued in The next Page

52-a - The rest:

left to right

assume $x \leq n$

having $\lceil x \rceil - 1 < x$ we have

$\lceil x \rceil - 1 < x \leq n \rightarrow \lceil x \rceil - 1 < n$

$\rightarrow \lceil x \rceil < n+1 \rightarrow \lceil x \rceil \leq n$

b) assume $n \leq x$. we know $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$

left to right

having $x < \lfloor x \rfloor + 1$ and $n \leq x$ we have

$n \leq x < \lfloor x \rfloor + 1 \rightarrow n < \lfloor x \rfloor + 1 \rightarrow n \leq \lfloor x \rfloor$

right to left

assume $n \leq \lfloor x \rfloor$

having $\lfloor x \rfloor \leq x$ we have $n \leq \lfloor x \rfloor \leq x$

Thus $n \leq x$

(10) $f: Y \to Z$ and $g: X \to Y$

f and g are inversible

Note: The fact That $f \circ g$ is inversible is not trivial and needs to be shown. Since The question assumes it implicitly you can ignore Proving That $f \circ g$ is inversible. But here is The Proof:

$f \circ g$ is inversible:

$$(f \circ g)(x_1) = (f \circ g)(x_2) \to f(\overbrace{g(x_1)}^{y_1}) = f(\overbrace{g(x_2)}^{y_2})$$

Since f is 1-1 we have $y_1 = y_2$ so

$g(x_1) = g(x_2)$ and since g is 1-1 we have $x_1 = x_2$

This shows $f \circ g$ is 1-1.

take a z in Z. Since f is onto $\exists y \in Y$ s.t. $f(y) = z$. call it y^*. since g is onto $\exists x \in X$ s.t $g(x) = y^*$. call it x^*. $f \circ g(x^*) = f(g(x^*)) = f(y^*) = z$

So we can find an $x \in X$ for any arbitrary $z \in Z$

That means $f \circ g$ is onto.

$f \circ g$ is 1-1 & onto $\to$ $f \circ g$ is inversible.

Next Page

(10) The rest :

to show that two functions are equal we need to show 2 things :

1. Their domain are the same
2. They work the same on this common domain.

Both $(f \circ g)^{-1}$ and $g^{-1} \circ f^{-1}$ have Z as their domain so 1 is proved.

for 2, let's say $(f \circ g)^{-1}$ assigns x to an arbitrary element $z \in Z$ which means $(f \circ g)^{-1}(z) = x$

if $g^{-1} \circ f^{-1}$ also assigns the same x to this z they are working the same on their domain.

$(f \circ g)^{-1}(z) = x \rightarrow (f \circ g)(x) = z \rightarrow f(g(x)) = z$

$\rightarrow f^{-1}(z) = g(x)$ ✡

now let's see if $g^{-1} \circ f^{-1}(z)$ is x or not :

$$g^{-1} \circ f^{-1}(z) = g^{-1}(f^{-1}(z)) \overset{✡}{=} g^{-1}(g(x)) = x$$

∎

(12) $f: A \longrightarrow B \qquad |A| = |B|$

f is 1-1 $\longleftrightarrow$ f is onto

let's say $|A| = |B| = n$ and $\{x_1, x_2, \dots, x_n\} = A$

and $\{y_1, y_2 \cdots y_n\} = B$

// assuming That f is onto we show f is 1-1:

WLG $f(x_1) = f(x_2) = y_1$ and for $i \geq 3$ $f(x_i) = y_i$

That means y_2 is not assigned to any thing which Contradicts The Fact That f is onto Therefor one of x_1 or x_2 must be assigned to y_2 which ends in f to be 1-1.

// assuming That f is ~~on~~ 1-1

if f is not onto There would be a y_i in B That is not assigned to anything WLG say That y_1 is That member $n-1$ other members of B are assigned to n member of A and since f is 1-1 according to Pigean Principal There would be one member of A That is not assigned to any member of B which Contradicts the Fact That f is a Function. So f is onto.

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a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ Yes

$\lceil x \rceil = y \longrightarrow y \in \mathbb{Z} \longrightarrow \lfloor y \rfloor = y \rightarrow \lfloor \lceil x \rceil \rfloor = y = \lceil x \rceil$

b) $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ No

Counterexample $\begin{matrix} x = 2.7 \\ y = 2.7 \end{matrix} \rightarrow \lfloor 5.4 \rfloor \neq \lfloor 2.7 \rfloor + \lfloor 2.7 \rfloor$

$5 \neq 2+2$

c) $\left\lceil \frac{\lceil x/2 \rceil}{2} \right\rceil = \lceil x/4 \rceil$ Yes

$\lceil x/2 \rceil = y \longrightarrow y - 1 < x/2 \leq y \longrightarrow \frac{y-1}{2} < \frac{x}{4} \leq y/2$

$\longrightarrow \frac{y}{2} - 1 < \frac{y}{2} - \frac{1}{2} < \frac{x}{4} \leq \frac{y}{2} \rightarrow \frac{y}{2} - 1 < \frac{x}{4} \leq \frac{y}{2}$ ①

$\frac{y}{2} - \frac{1}{2} < x/4 \leq y/2$ ②

if y is even according to ①.

and if is odd according to ② we have $\lceil x/4 \rceil = \lceil y/2 \rceil$

subtituting y we have $\lceil x/4 \rceil = \left\lceil \frac{\lceil x/2 \rceil}{2} \right\rceil$

d) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ NO $x = 3.5 \rightarrow 2 \neq 1$

e) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$ Yes

$x = n + \varepsilon_1$, $y = m + \varepsilon_2$ Consider ~~three~~ 4 cases

① $0 \leq \varepsilon_1, \varepsilon_2 < 1/2$

② $0 \leq \varepsilon_1 < 1/2$ & $1/2 \leq \varepsilon_2 < 1$ & $\varepsilon_1 + \varepsilon_2 \geq 1$ WLG

③ $1/2 \leq \varepsilon_1, \varepsilon_2 < 1$

④ $0 \leq \varepsilon_1 < 1/2$ & $1/2 \leq \varepsilon_2 < 1$ & $\varepsilon_1 + \varepsilon_2 < 1$

①: left = right = $2x + 2y$

③: left = $2x + 2y + 1$, right = $2x + 2y + 2 \rightarrow$ left < right