8:45

University of Puerto Rico Mayagüez
Department of Electrical and Computer Engineering
ICOM 4075: Foundations of Computing

Partial Exam I (100 Points)

Question1 – (40 points) Short questions (2 points each)

1- Define the scope of a quantifier.

Part of an assertion in which Variables are bound by The quantifier. $\forall x\ (F(x) \wedge S(x))$ x: wide scope
$\forall x\ F(x) \wedge \forall x\ S(x)$ x: narrow scope

2- What is the truth set of the following predicate:

$x^2 + 1 \leq 0$ *where the domain is R*

$\emptyset$

3- What is the inverse of the following implication?

I go to school only if it's not raining.

If I don't go to school Then it is raining.

4- What is the converse of the following conditional proposition?

$a^2 = b^2$ *unless* $a \neq b$

$a^2 = b^2 \longrightarrow a = b$

5- What is the contrapositive of the following implication?

Snowing is sufficient for me to go fishing.

If I don't go Fishing Then It's not snowing.

6- If P(x,y,z) is "z is an equation with x and y as its solutions" find two sets of domain for the following statement such that for one of them the statement is true and for the other one is false:

$D_T = \{D_z, D_{x,y}\}$ $D_F = \{D'_z, D'_{x,y}\}$

$\exists z\ \forall x \forall y\ \ \neg p(x, y, z)$

D_z = Set of all equations on $\mathbb{R}$ $D_{x,y} = \mathbb{R}$ $D'_z = \{x+y=0\}$ $D'_{x,y} = \mathbb{Z}$

7- Translate the following statement into the logical expression (using quantifiers and predicates) in two ways; one with the domain as people and the other with the domain as students:

Some students in your class do not want to be rich Domain students: $\exists x\ C(x) \wedge \neg R(x)$

C(X): x is in class
R(x): x wants to be rich
S(x): x is student
SC(x): x is student in class

Domain People
Answer1: $\exists x\ SC(x) \wedge \neg R(x)$
Answer2: $\exists x\ S(x) \wedge C(x) \wedge \neg R($

8- Negate the following proposition such that no negate sign comes before a quantifier:

$\forall x\ \exists y\ (p(x, y) \rightarrow\ \forall z\ (q(x, z, 10)\ \oplus\ q(x, z, 11)))$

$\exists x\ \forall y\ \ (p(x,y) \wedge \exists z\ \neg\ (q(x,z,10) \oplus q(x,z,11)))$

9- Parenthesize: $\left(\left(p \rightarrow (q \vee r)\right) \leftrightarrow \left((\neg s \wedge p) \rightarrow (\neg q \wedge r)\right)\right)$

Translate the statements of questions 10 to 16 assuming that L(x,y) is "x loves y" and H(x,y) is "x hates y" and the domain for both x and y are all people.

10- $\exists x \exists y \exists z\ (x \neq y) \wedge L(z,x) \wedge H(z,y)$

There are two different persons x, y and a person z That z loves x and hates y.

11- $\exists x\ L(x, Juan) \wedge H(x, Hector)$

There is a person who loves Juan and hates Hector.

12- $\exists x\ L(x, Juan) \wedge (\forall z\ L(z, Juan) \rightarrow z = x)$

There is a unique person who loves Juan.

13- $L(Mary, Juan) \wedge L(Sarah, Juan) \rightarrow H(Mary, Sarah) \vee H(Sarah, Mary)$

If both Mary and Sarah Love Juan, either Mary hates Sarah or sarah hates Mary.

14- $\forall x\ \exists y\ \forall z\ L(x,z) \rightarrow H(y,z)$

For any person x, There are some people y such That whoever x loves, y hates.

15- There is one and only one person that if you love him he loves you too.

$\exists! x \quad L(you, x) \longrightarrow L(x, you)$

16- There is at least one person who if Juan doesn't love her, she would hate Juan.

$\exists x \quad \neg L(Juan, x) \longrightarrow H(x, Juan)$

Multiple choice questions from 17 to 20:

17- Which one is not tautology?

a) $\forall x\ P(x) \rightarrow P(x) \vee Q(x)$
b) $(p \rightarrow q) \wedge (p \rightarrow \neg q)$
c) $P(x)\, for\ arbitrary\ x\ \rightarrow\ \forall x\ P(x)$
d) $\neg \exists x P(x) \leftrightarrow \forall x\ \neg P(x)$

18- In an exam one question was: "prove that there are no solutions in positive integers for the equation $x^3 + y^3 = 21$". One student wrote the following: "if $x^3 + y^3 = 21$ since 3^3=27 then x and y must be less than 3. Thus the possibilities for x and y are 1 or 2. But none of 1^3+2^3, 1^3+1^3, 2^3+2^3 makes 21".

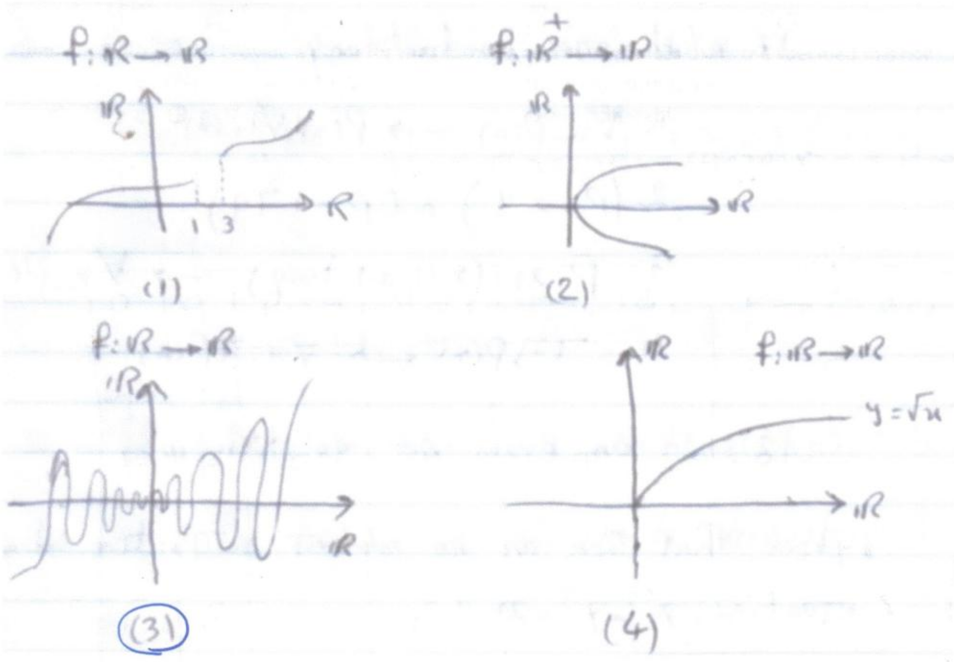
a) This is a proof by example
b) This is proof by cases
c) This is a disproof by example
d) This is a wrong proof

excluded ~~19~~- Which one is not a tautology?

a) $(p \to q) \leftrightarrow (q \vee \neg p)$ arbitrary
b) $p \wedge q \to p \vee q$
c) $\exists x\, P(x) \to P(a)\, for\ a \in U\ where\ U\ is\ the\ domain$
d) $\exists x \left(P(x) \vee Q(x)\right) \leftrightarrow \exists x\, P(x) \vee \exists x\, Q(x)$

20- Which one is a function (Place an X next to it):

20- which one is a function



1: $\exists x \neg P(x)$ — Premise

2: $\neg P(a)$ — EI

3: $\forall x\ P(x) \vee Q(x)$ — Premise

4: $P(a) \vee Q(a)$ — UI (in Partiqular for a)

Question2- (10 points) Show that the following argument is valid:

$\forall x \quad P(x) \vee Q(x)$
$\forall x \quad \neg Q(x) \vee S(x)$
$\forall y \quad R(y) \rightarrow \neg S(y)$
$\exists x \quad \neg P(x)$

$\therefore \quad \exists x \quad \neg R(x)$

5: $Q(a)$ — 2 & 4

6: $\forall x\ \neg Q(x) \vee S(x)$ — Premise

7: $\neg Q(a) \vee S(a)$ — UI (in Partiqular for a)

8: $S(a)$ — 5 & 7

9: $\forall y\ R(y) \rightarrow \neg S(y)$ — Premise

10: $R(a) \rightarrow \neg S(a)$ — UI (in Partiqular for a)

11: $S(a) \rightarrow \neg R(a)$ — Contrapositive of 10

12: $\neg R(a)$ — 8 & 11 & MP

13: $\exists x \quad \neg R(x)$ — EG

together with inference rules, equivalences are allowed also.

Can be inferred directly from 8 & 10 & MT

Question 3-(10 points) Provide a proof for the following two equivalences using the specified method:

a) $(p \rightarrow q) \wedge \neg p \ \equiv \ p \rightarrow (q \wedge \neg p)$ — Not using truth table

b) $(p \rightarrow q) \oplus p \equiv p \rightarrow (q \oplus p)$ — Using truth table

a) $(p \rightarrow q) \wedge \neg p \equiv (q \vee \neg p) \wedge \neg p \equiv (q \wedge \neg p) \vee (\neg p \wedge \neg p) \equiv (q \wedge \neg p) \vee \neg p \equiv p \rightarrow (q \wedge \neg p)$

b)

p	q	$p \rightarrow q$	$q \oplus p$	$(p \rightarrow q) \oplus p$	$p \rightarrow (q \oplus p)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	F	T	T

Question 4*-(10 points)* Prove that any 2-digit integer number ab satisfies the following property:

ab-ba = 9k

(Note: for example in a 2-digit number like 64, a=6 and b=4. Hint: Since the number is in base 10, try to use the representation of numbers in base 10)

What kind of proof did you use?

$$ab = b + 10a$$
$$ba = a + 10b$$

$$ab - ba = b + 10a - a - 10b = 9a - 9b = 9(a-b)$$

Direct proof

Question 5*-(10 points)* If $f: A \to B$ and $g: B \to C$ are functions and f and g are 1-1 and onto, show that $g \circ f$ is a bijection.

$$g \circ f(x_1) = g \circ f(x_2) \to g(\underbrace{f(x_1)}_{y_1}) = g(\underbrace{f(x_2)}_{y_2}) \to g(y_1) = g(y_2)$$

$$(g \text{ is } 1\text{-}1) \wedge (g(y_1) = g(y_2)) \to y_1 = y_2 \to f(x_1) = f(x_2)$$

$$(f \text{ is } 1\text{-}1) \text{ Thus } \quad (f(x_1) = f(x_2) \to x_1 = x_2)$$

Take an arbitrary element $c \in C$. Since g is onto, There exist an element $b \in B$ s.t. $g(b) = c$. For $b \in B$ there exist $a \in A$ s.t. $f(a) = b$ because f is onto.

$$g \circ f(a) = g(f(a)) = g(b) = c$$

That means for any $c \in C$ There exist an $a \in A$ s.t. $f \circ g(a) = c$. Therefor $f \circ g$ is onto.

Question 5- (10 points) Proof the followings set propositions:

a) $A \times B = \emptyset \rightarrow A = \emptyset \vee B = \emptyset$ Using contradiction

b) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ Using 2-way subsets

c) $(A \cap B) \cup (A \cap B^c) = A$ Using identity laws

a) $\neg(A=\emptyset \vee B=\emptyset) \equiv A \neq \emptyset \wedge B \neq \emptyset$

$A \neq \emptyset \rightarrow \exists a \in A$
$B \neq \emptyset \rightarrow \exists b \in B$
$\} \rightarrow (a,b) \in A \times B$ by The definition of $A \times B$

$\rightarrow A \times B \neq \emptyset \equiv \neg(A \times B = \emptyset)$

b) $x \in (A \cup B)-(A \cap B) \rightarrow x \in A \cup B \wedge x \notin A \cap B \rightarrow x \in A \cup B \wedge x \in \overline{A \cap B}$

$\rightarrow x \in A \cup B \wedge x \in \bar{A} \cup \bar{B} \rightarrow (x \in A \vee x \in B) \wedge (x \in \bar{A} \vee x \in \bar{B})$

$\rightarrow ((x \in A \vee x \in B) \wedge x \in \bar{A}) \vee ((x \in A \vee x \in B) \wedge x \in \bar{B})$

$\rightarrow (x \in A \wedge x \in \bar{A}) \vee (x \in B \wedge x \in \bar{A}) \vee (x \in A \wedge x \in \bar{B}) \vee (x \in B \wedge x \in \bar{B})$

$\rightarrow x \in A \cap \bar{A} \vee x \in B \cap \bar{A} \vee x \in A \cap \bar{B} \vee x \in B \cap \bar{B} \rightarrow F \vee x \in B-A \vee x \in A-B \vee F$

$\rightarrow x \in B-A \vee x \in A-B \rightarrow x \in (B-A) \cup (A-B)$

Therefore $(A \cup B)-(A \cap B) \subseteq (A-B) \cup (B-A)$

The other side:

$x \in (B-A) \cup (A-B) \rightarrow x \in B-A \vee x \in A-B \rightarrow (x \in B \wedge x \notin A) \vee (x \in A \wedge x \notin B)$

$\rightarrow ((x \in B \wedge x \notin A) \vee x \in A) \wedge ((x \in B \wedge x \notin A) \vee x \notin B) \rightarrow (x \in B \vee x \in A) \wedge (x \notin A \vee x \in A) \wedge$

$(x \in B \vee x \notin B) \wedge (x \notin A \vee x \notin B) \rightarrow x \in B \cup A \wedge (x \in \bar{A} \vee x \in A) \wedge (x \in B \vee x \in \bar{B}) \wedge (x \in \bar{A} \vee x \in \bar{B})$

$\rightarrow x \in B \cup A \wedge x \in A \cup \bar{A} \wedge x \in B \cup \bar{B} \wedge x \in \bar{A} \cup \bar{B} \rightarrow x \in B \cup A \wedge T \wedge T \wedge x \in \bar{A} \cup \bar{B} \rightarrow$

$x \in B \cup A \wedge x \in \overline{A \cap B} \rightarrow x \in B \cup A \wedge x \notin A \cap B \rightarrow x \in (B \cup A)-(A \cap B)$

Therefor $(B-A) \cup (A-B) \subseteq (A \cup B)-(A \cap B)$

c) $(A \cap B) \cup (A \cap \bar{B}) = A \cap (B \cup \bar{B}) = A \cap U = A$

Question 7- *(10 points)* Show that if $x\epsilon R$ and $n\epsilon Z$ then $x < n \leftrightarrow \lfloor x \rfloor < n$

we have the property

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

assume $x < n$

$$\lfloor x \rfloor \leq x \wedge x < n \rightarrow \lfloor x \rfloor < n$$

The other side : assume $\lfloor x \rfloor < n$

$$\lfloor x \rfloor < n \rightarrow \lfloor x \rfloor \leq n-1$$

$$x < \lfloor x \rfloor + 1 \wedge \lfloor x \rfloor \leq n-1 \rightarrow x < n-1+1$$

$$\rightarrow x < n$$

Question 8- (BONUS 10 points)- If $f: A \rightarrow B$ is a function and $S \subseteq A, T \subseteq A$ and $L \subseteq B$ show that:

a) $f(S \cap T) \subseteq f(S) \cap f(T)$

b) $f^{-1}(L^c) = (f^{-1}(L))^c$

a) $y \in f(S \cap T) \rightarrow \exists x \in S \cap T \; ; \; f(x) = y$

$\rightarrow (x \in S \wedge f(x) = y) \wedge (x \in T \wedge f(x) = y)$

$\rightarrow y \in f(S) \wedge y \in f(T) \rightarrow y \in f(S) \cap f(T)$

b) $f^{-1}(\bar{L}) = \{x \in A \mid f(x) \in \bar{L}\} = \{x \in A \mid f(x) \notin L\}$

$= \{x \in A \mid x \notin f^{-1}(L)\} = \{x \in A \mid x \in \overline{f^{-1}(L)}\}$

Note: for part a, it is not necessary for f to be inversible. But if you give a proof in which you assume That f is inversible, We accept it.