University of Puerto Rico Mayagüez
Department of Electrical and Computer Engineering
ICOM 4075: Foundations of Computing

Partial Exam 3 (100 Points)

Name: CLAVE

Section:

Instructor: B. Vélez

You will have 120 minutes to answer as many questions as possible. Any points that you obtain beyond 100 will be considered bonus points that you can accumulate towards improving your grades in previous partial exams.

Question 1 (20 points). Use mathematical induction to prove that for any integer $n \geq 1$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(5 points) Basis Step: $n=1$

$$\sum_{i=1}^{\cancel{n}\,1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$\frac{\cancel{n}\,1}{\cancel{n}+1} = \frac{1}{1+1} = \frac{1}{2}$$

True.

(5 points) Inductive Hypothesis:

For some arbitrary $k \geq 1$ $\quad \displaystyle\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$

(10 points) Inductive Step:

$$\sum_{i=1}^{k+1} \left(\frac{1}{i(i+1)}\right) = \frac{1}{(k+1)(k+2)} + \boxed{\sum_{i=1}^{k} \frac{1}{i(i+1)}}$$

Ind. Hyp.

$$= \frac{1}{(k+1)(k+2)} + \boxed{\frac{k}{(k+1)}}$$

$$= \frac{1+k(k+2)}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^{\cancel{2}\,1}}{\cancel{(k+1)}(k+2)} = \frac{k+1}{(k+1)+1}$$

Q.E.D.

Question 2 (20 points). Prove using the Principle of Mathematical Induction that for any positive n:

$$\frac{1}{2n} \le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \prod_{i=1}^{n} \frac{2i-1}{2i}$$

(5 points) Basis Step: $n = 1$

$$\frac{1}{2(1)} = \frac{1}{2}$$

$$\prod_{i=1}^{1} \frac{2i-1}{2i} = \frac{1}{2}$$

True.

(5 points) Inductive Hypothesis:

For an arbitrary $k \geq 1$ $\quad \frac{1}{2k} \le \prod_{i=1}^{k} \frac{2i-1}{2i}$

(10 points) Inductive Step:

Must show: $\frac{1}{2(k+1)} \le \prod_{i=1}^{k+1} \frac{2i-1}{2i}$

Ind. Hyp.

$$\prod_{i=1}^{k+1} \frac{2i-1}{2i} = \left(\prod_{i=1}^{k} \frac{2i-1}{2i} \right) \cdot \frac{2(k+1)-1}{2(k+1)} \geq \left(\frac{1}{2k} \right) \frac{2k+1}{2k+2}$$

$$= \frac{1}{2(k+1)} \cdot \left(\frac{2k+1}{2k} \right) > \frac{1}{2(k+1)} \cdot 1 = \frac{1}{2(k+1)}$$

> 1

Q.E.D.

Question 3 (20 points). Use **strong** mathematical induction to prove that for any $n \geq 3$ we can write n as a linear combination of 2 and 5. That is,

$n = 2t + 5s$ for some non-negative integers s and t

(5 points) Basis Step:

For $n = 4$ True since $4 = 2(2) + 5(0)$

$n = 5$ True since $5 = 2(0) + 5(1)$

(5 points) Inductive Hypothesis: Let $P(n)$: $n = 2t + 5s$ for some ints t & s.

For an arbitrary $k \geq 5$, $P(5) \wedge P(6) \wedge \ldots \wedge P(k)$

(10 points) Inductive Step:

Must show $k+1$ can be written as linear combination of 2 & 5.

By Inductive Hypothesis $(k-1) = 2(t) + 5(s)$ for some t, s non-negative integers.

Then $\underbrace{2(t) + 5(s)}_{k-1} + 2 = 2(t+1) + 5(s)$

$= (k-1) + 2$

$= k+1$

Thus $(k+1)$ must also be written as a linear combination of s & t.

Q.E.D

Question 4a (15 points). Consider the recursive definition of the set S of all pairs of non-negative integers of the form (a, b):

Basis Step: $(0,0) \in S$

Recursive Step: if $(a,b) \in S$ **then** $(a+4, b+3) \in S$ **and** $(a+3, b+4) \in S$

Prove by structural induction that for any $(a,b) \in S,\ 7 \mid (a+b)$

(2 points) Basis Step:

$(a,b) = (0,0) \qquad a+b = \emptyset \quad 7 \mid \emptyset \quad$ True.

(3 points) Inductive Hypothesis:

For an arbitrary $(a,b) \in S \quad 7 \mid a+b$. or $(a+b) = 7k$ for some k.

(10 points) Inductive Step:

Two cases:

Case 1 $(a+4, b+3) \quad (a+4)+(b+3) = (a+b)+7 = 7k+7 = 7(k+1)$ True.

Case 2: $(a+3, b+4) \quad (a+3)+(b+4) = (a+b)+7 = 7k+7 = 7(k+1)$ True

Q.E.D.

Question 4b (5 points). Prove that S is not equal to the set of all pairs (a,b) such that $7 \mid (a+b)$

It suffices to find an example of an pair (a,b) such that $7 \mid a+b$, but $(a,b) \notin S$. $(5,2)$ is such a pair since $7 \mid 5+2$ but $(5,2)$ cannot be generated by any of the rules. Both cases will require another pair (a,b) s.t. $b+3=2$ or $b+4=2$. Impossible since $b \geq 0$.

Question 5a (15 points). Use a similar approach to Algorithm 5 from the textbook (see below) to design a worst-case $O(n)$ comparisons <u>recursive algorithm</u> that returns `true` if a list of integers $a_1 \dots a_n$ contains at least one negative number and returns `false` otherwise. The algorithm should be best-case $O(1)$. Non-recursive responses will receive no credit.

ALGORITHM 5 A Recursive Linear Search Algorithm.

procedure *search*$(i, j, x$: i, j, x integers, $1 \le i \le j \le n)$
if $a_i = x$ **then**
    return i
else if $i = j$ **then**
    return 0
else
    return *search*$(i + 1, j, x)$
{output is the location of x in $a_1, a_2, \dots, a_n$ if it appears; otherwise it is 0}

```
procedure hasNegative(a₁, a₂, …, aₙ, i, ~~j~~: integers)
    if i=n then return (aᵢ<0)
    return (aᵢ<0) or hasNegative(a₁, … aₙ, i+1)
```

{ output is true if the list $a_1 \dots a_n$ contains a negative number between locations i and n. False otherwise. }

Question 5b (5 points). Explain why your algorithm does O(n) comparisons in the worst case:

For any $n \geq 1$ the algorithm will induce recursive calls for $i = 1, 2, \dots, n$, a total of n calls. Each call does 2 comparisons ($i = n$ & $a_i < 0$), therefore total comparisons is $2n \in \Theta(n)$

Question 6 (20 points). Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute the product $r \cdot s$. Show that no matter how you split the piles, the sum of the products computed at each step equals:

Let. $P(n)$: $\sum r_i \cdot s_i = \frac{n(n-1)}{2}$

Hint: Use strong induction

We will use strong induction on n, the number of stones in the first pile.

Basis We could potentially work from basic cases of 1 or Ø stones, but for simplicity we choose to begin with 2 stones.

$n=2$ In this case there is only one way of splitting; two piles of one stone. $r=1$ and $s=1$. Since no more splitting is necessary $\sum r \cdot s = 1 \cdot 1 = 1$.

Since $\frac{n(n-1)}{2} = \frac{2(2-1)}{2} = \frac{2}{2} = 1$ The basis is true.

Strong Inductive Hypothesis: For an arbitrary $k \geq 2$ any pile of size $2 \leq k' \leq k$ will induce a sum of products $\frac{k'(k'-1)}{2}$

Inductive Step: Consider a pile of size $(k+1)$ splitted into 2 piles of size r and s, that is $r+s = k+1$.

$$\text{Sum of products}_{k+1} = \text{Sum of Products}_r + \text{SumOfProducts}_s + r \cdot s$$

$$= \frac{r(r-1)}{2} + \frac{s(s-1)}{2} + \frac{2r \cdot s}{2} = \frac{r^2 - r + s^2 - s + 2rs}{2}$$

$$= \frac{(r^2 + 2rs + s^2) - (r+s)}{2} = \frac{(r+s)^2 - (r+s)}{2} = \frac{(k+1)^2 - (k+1)}{2}$$

$$= \frac{(k+1)(k+1-1)}{2} = \frac{(k+1)k}{2} \quad \text{Q.E.D.}$$

Problem	Points
1	/20
2	/20
3	/20
4	/20
5	/20
TOTAL	/100
BONUS	/20